CS482:
Monte Carlo Integration

Sung-Eui Yoon
(윤성의)

http://sglab.kaist.ac.kr/~sungeui/ICG

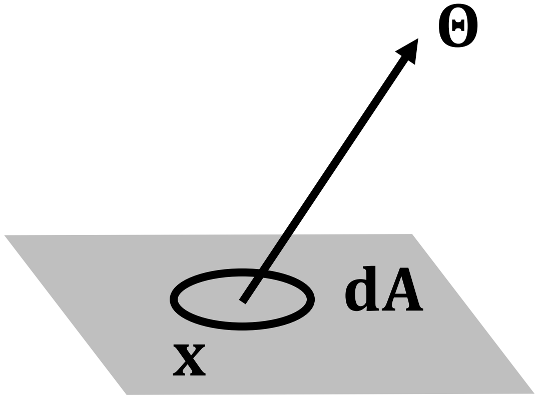
KAIST

Class Objectives (Ch. 14)

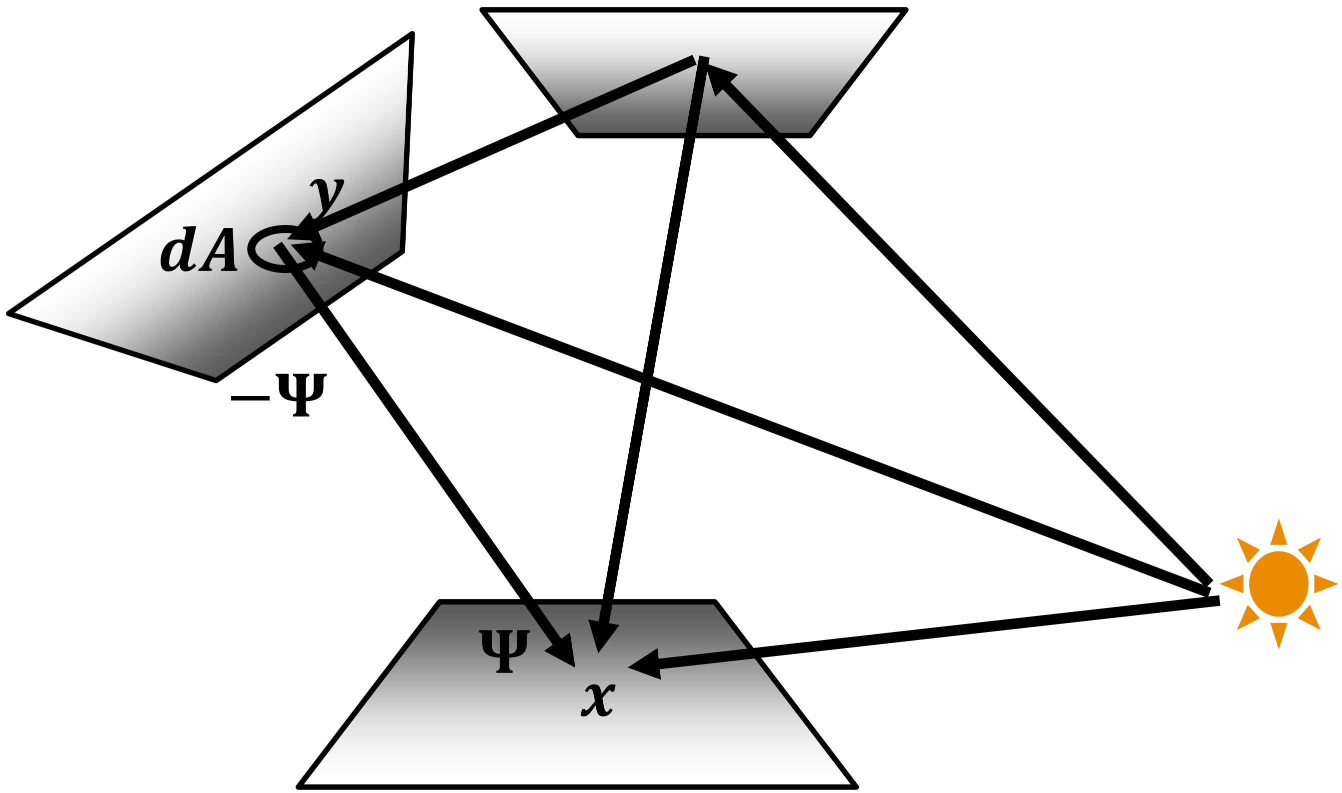
- Sampling approach for solving the rendering equation
  - Monte Carlo integration
  - Estimator and its variance

Radiance Evaluation

- Fundamental problem in GI algorithm
  - Evaluate radiance at a given surface point in a given direction
  - Invariance defines radiance everywhere else



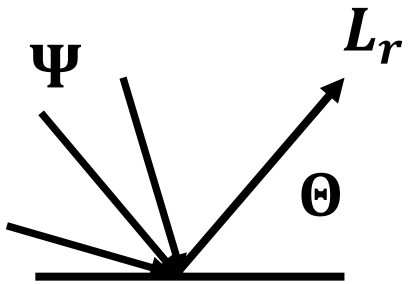
We need to find many paths…



Why Monte Carlo?

- **Radiance is hard to evaluate**

$$L_r(x \to \Theta) = \int_{\Psi} L(x \leftarrow \Psi) f_r(x, \Psi \to \Theta) \cos\theta_x dw_{\Psi},$$



- **Sample many paths**
  - **Integrate over all incoming directions**
- **Analytical integration is difficult**
  - **Need numerical techniques**

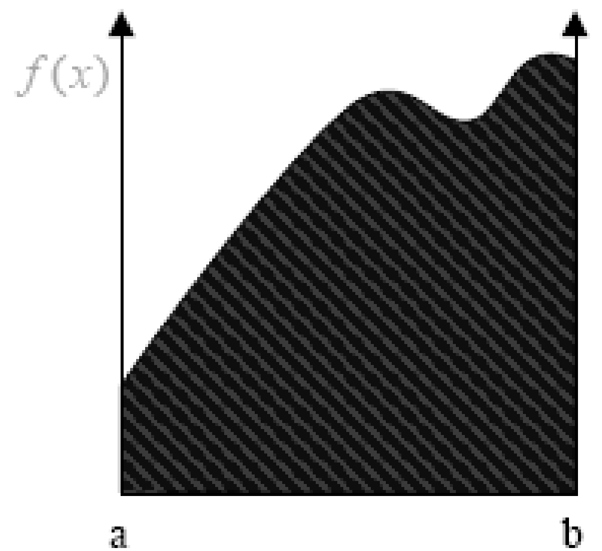
Monte Carlo Integration

- Numerical tool to evaluate integrals
  - Use sampling
- Stochastic errors
- Unbiased
  - On average, we get the right answer

Numerical Integration

- A one-dimensional integral:

$$I = \int_a^b f(x)dx$$

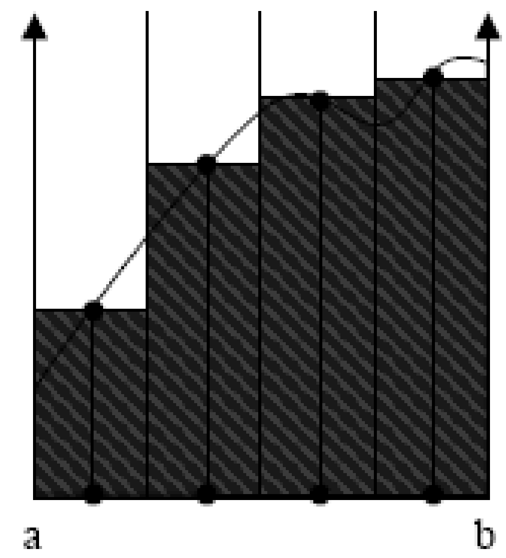


Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x)dx$$

$$\approx \sum_{i=1}^{N} w_i f(x_i)$$

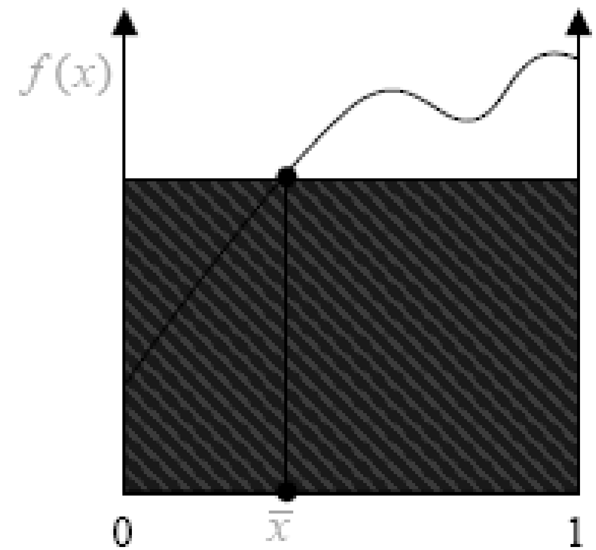


Monte Carlo Integration

Primary estimator:

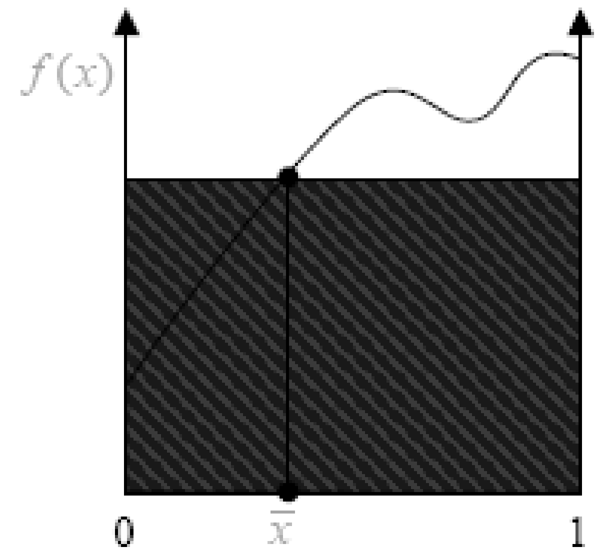
$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})$$



Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})$$

$$E(I_{prim}) = \int_0^1 f(x)p(x)dx = \int_0^1 f(x)1dx = I$$

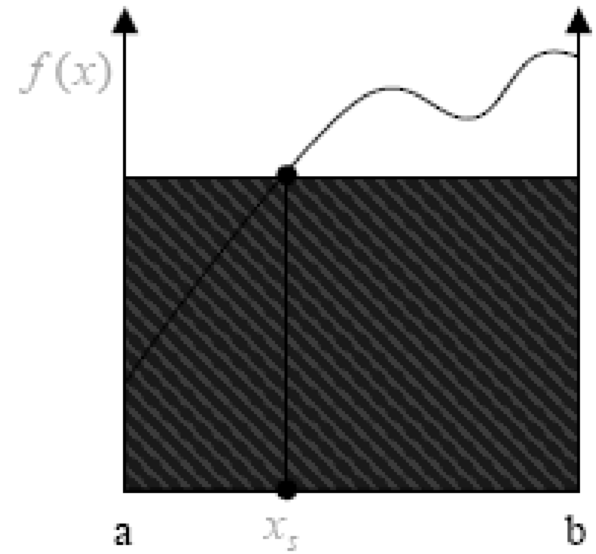
Unbiased estimator!

Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(x_s)(b-a)$$



$$E(I_{prim}) = \int_a^b f(x)(b-a)\,p(x)dx = \int_a^b f(x)(b-a)\frac{1}{(b-a)}dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator → a measure of the stochastic error

$$\sigma^2_{prim} = \int_a^b \left[\frac{f(x)}{p(x)} - I\right]^2 p(x)dx$$

- **Consider p(x) for estimate**
- **We will study it as importance sampling later**

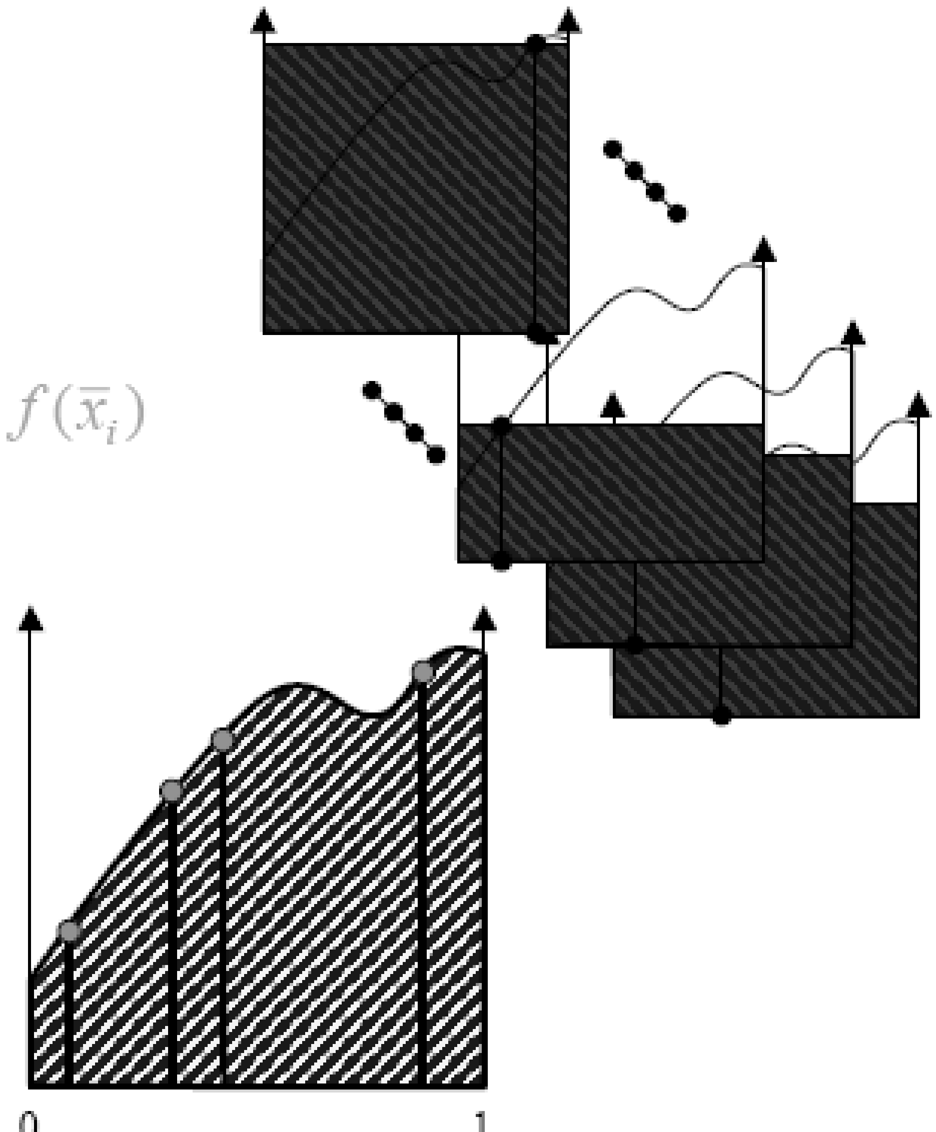
More samples

Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{sec} = \frac{1}{N}\sum_{i=1}^{N} f(\bar{x}_i)$

Variance $\sigma^2_{sec} = \sigma^2_{prim} / N$



Mean Square Error of MC Estimator

- **MSE**

$$MSE(\hat{Y}) = E[(\hat{Y} - Y)^2] = \frac{1}{N}\sum_i (\hat{Y}_i - Y_i)^2.$$

- **Decomposed into bias and variance terms**

$$\begin{aligned} MSE(\hat{Y}) &= E\left[\left(\hat{Y} - E[\hat{Y}]\right)^2\right] + \left(E(\hat{Y}) - Y\right)^2 \\ &= Var(\hat{Y}) + Bias(\hat{Y}, Y)^2. \end{aligned}$$

- **Bias: how far the estimation is away from the ground truth**
- **Variance: how far the estimation is away from its average estimator**

Bias of MC Estimator

$$
\begin{aligned}
E[\hat{I}] &= E\left[\frac{1}{N}\sum_i \frac{f(x_i)}{p(x_i)}\right] \\
&= \frac{1}{N}\int \sum_i \frac{f(x_i)}{p(x_i)} p(x)dx \\
&= \frac{1}{N}\sum_i \int \frac{f(x)}{p(x)} p(x)dx, \because x_i \text{ samples have the same } p(x) \\
&= \frac{N}{N}\int f(x)dx = I. \qquad (14.6)
\end{aligned}
$$

- **On average, it gives the right answer: unbiased**

KAIST

Variance of MC Estimator

$$Var(\hat{I}) = Var(\frac{1}{N}\sum_i \frac{f(x_i)}{p(x_i)})$$

$$= \frac{1}{N^2} Var(\sum_i \frac{f(x_i)}{p(x_i)})$$

$$= \frac{1}{N^2} \sum_i Var(\frac{f(x_i)}{p(x_i)}), \because x_i \text{ samples are independent from each other.}$$

$$= \frac{1}{N^2} N Var(\frac{f(x)}{p(x)}), \because x_i \text{ samples are from the same distribution.}$$

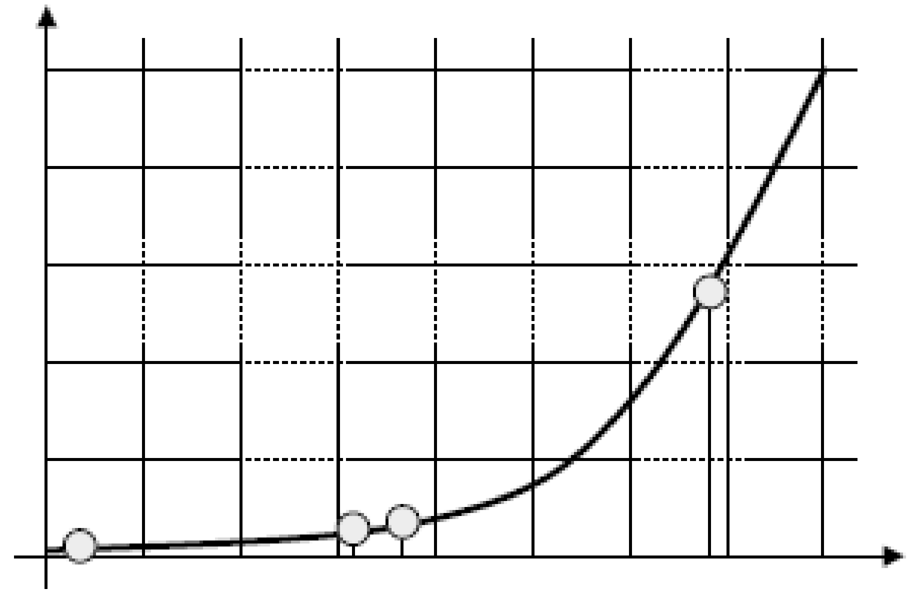
$$= \frac{1}{N} Var(\frac{f(x)}{p(x)}) = \frac{1}{N} \int \left(\frac{f(x)}{p(x)} - E\left[\frac{f(x)}{p(x)}\right]\right)^2 p(x)dx. \quad (14.7)$$

KAIST

MC Integration - Example

- Integral $I = \int_0^1 5x^4 dx = 1$

- Uniform sampling



- Samples :

$x_1 = .86$ $\langle I \rangle = 2.74$

$x_2 = .41$ $\langle I \rangle = 1.44$

$x_3 = .02$ $\langle I \rangle = 0.96$

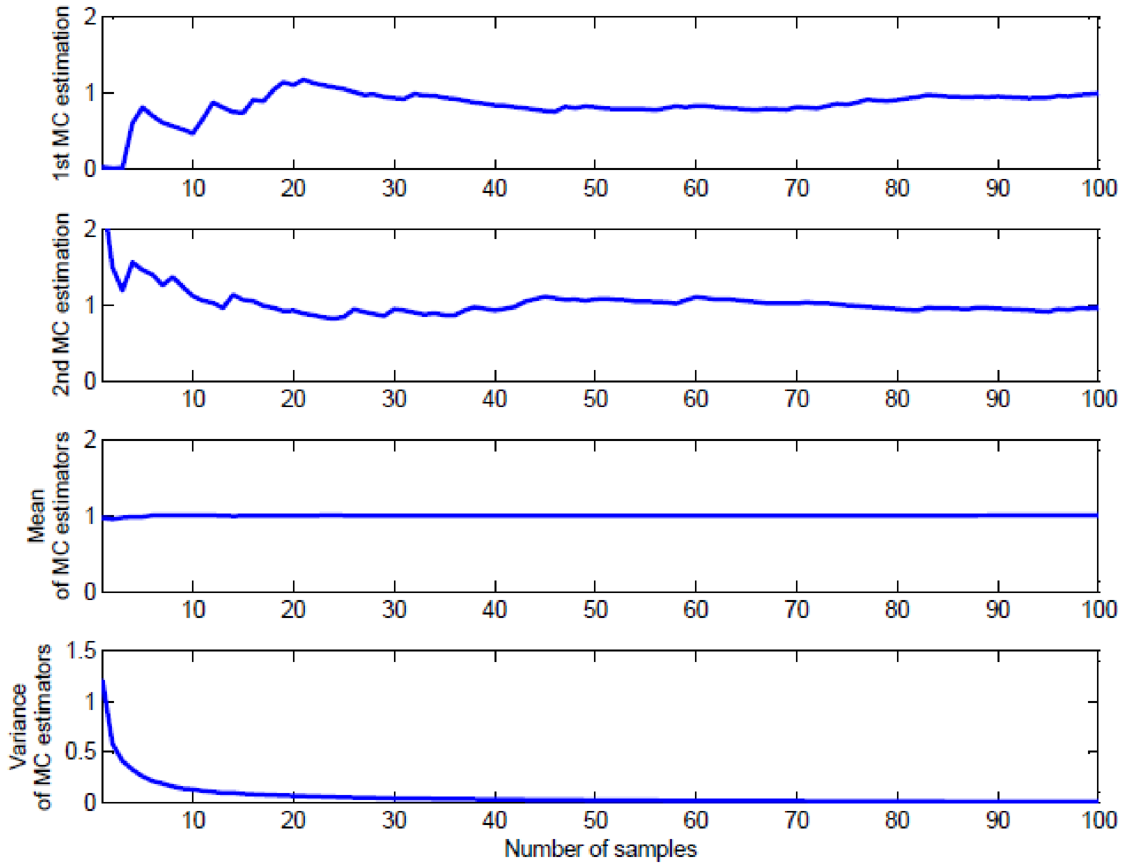
$x_4 = .38$ $\langle I \rangle = 0.75$

MC Integration - Example

- **Integral**

$$I = \int_0^1 4x^3 dx = 1$$

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} 4x_i^3,$$

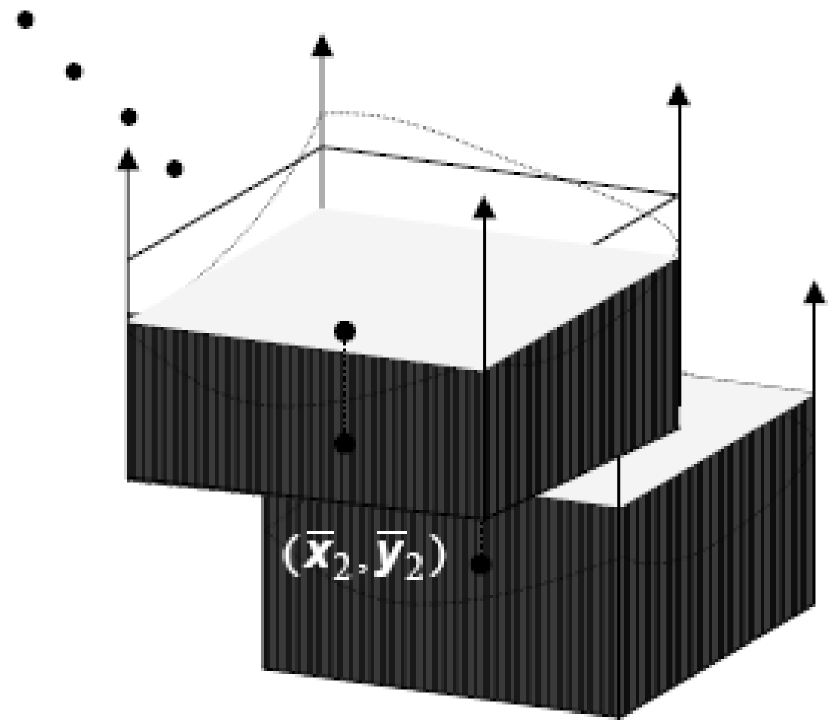


Code: mc_int_ex.m

MC Integration: 2D

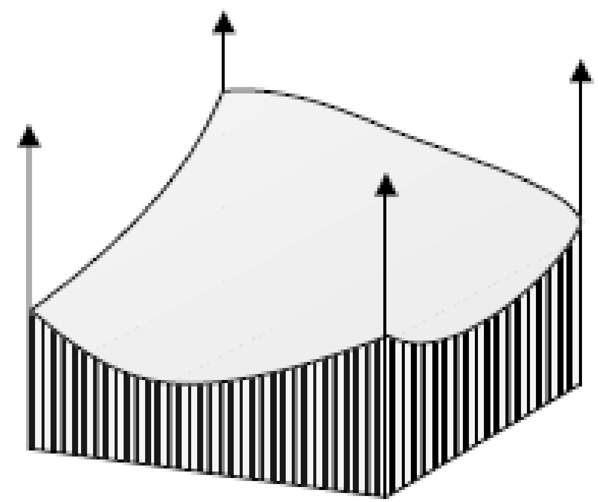
- Secondary estimator:

$$I_{\text{sec}} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(\overline{x}_i, \overline{y}_i)}{p(\overline{x}_i, \overline{y}_i)}$$



Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature



$$I = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

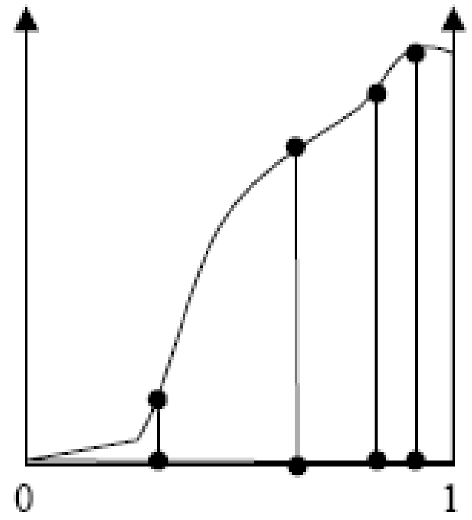
$$\langle I \rangle = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i, y_i)}{p(x_i, y_i)}$$

Advantages of MC

- **Convergence rate of** $O(\frac{1}{\sqrt{N}})$
- **Simple**
  - **Sampling**
  - **Point evaluation**
- **General**
  - **Works for high dimensions**
  - **Deals with discontinuities, crazy functions, etc.**

Importance Sampling

- Take more samples in important regions, where the function is large



From kavita's slides

Class Objectives (Ch. 14) were:

- **Sampling approach for solving the rendering equation**
  - **Monte Carlo integration**
  - **Estimator and its variance**

Next Time…

- Monte Carlo ray tracing

Homework

- Go over the next lecture slides before the class
- Watch 2 SIG/I3D/HPG videos and submit your summaries every Tue. class
  - Just one paragraph for each summary

Example:

Title: XXX XXXX XXXX

Abstract: this video is about accelerating the performance of ray tracing. To achieve its goal, they design a new technique for reordering rays, since by doing so, they can improve the ray coherence and thus improve the overall performance.

Any Questions?

- Submit four times in Sep./Oct.
- Come up with one question on what we have discussed in the class and submit at the end of the class
  - 1 for typical questions
  - 2 for questions that have some thoughts or surprise me

KAIST