CS580:
Monte Carlo Integration

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(윤성의)

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Questions

- **I was wondering how BSDFs and BSSRDFs are integrated into the rendering equation**

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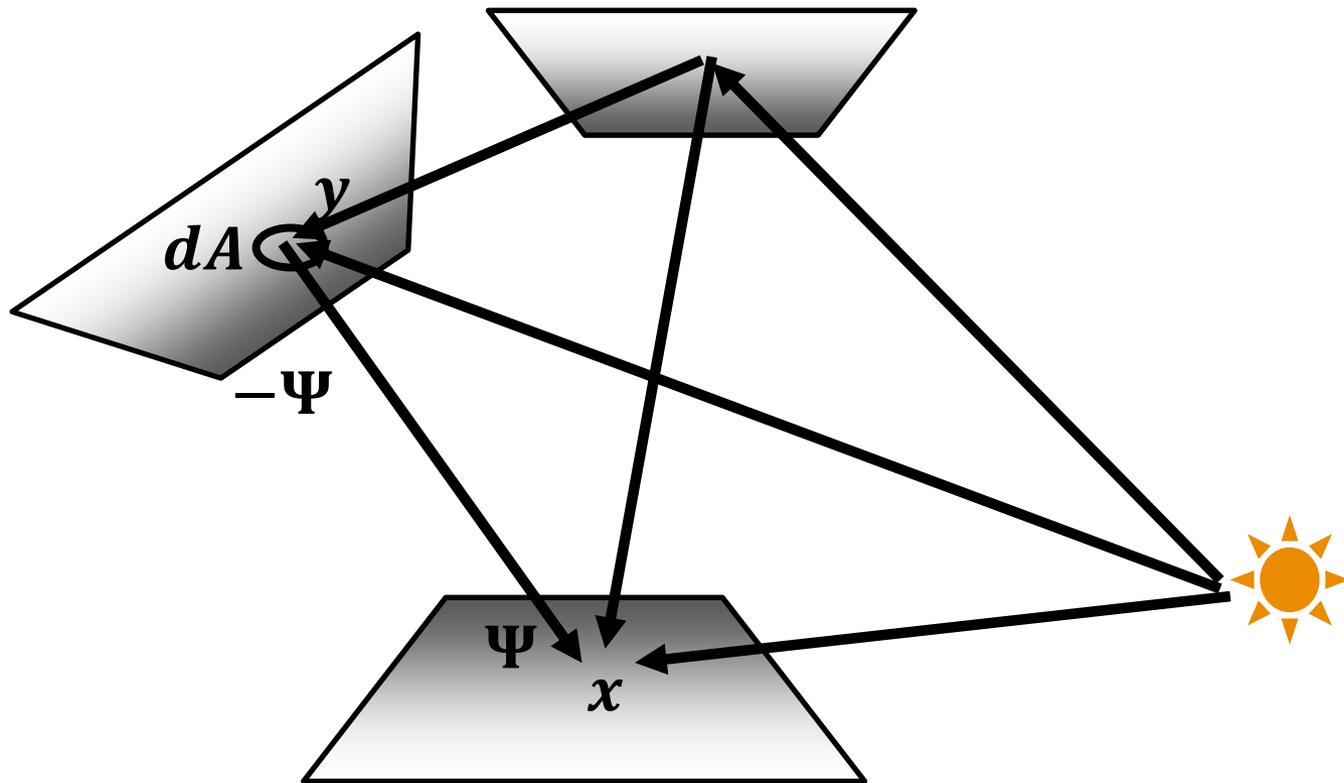
Class Objectives (Ch. 14)

- **Sampling approach for solving the rendering equation**
  - **Monte Carlo integration**
  - **Estimator and its variance**

Radiance Evaluation

- **Fundamental problem in GI algorithm**
    - **Evaluate radiance at a given surface point in a given direction**
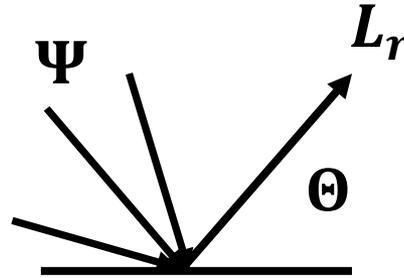   - **Invariance defines radiance everywhere else**
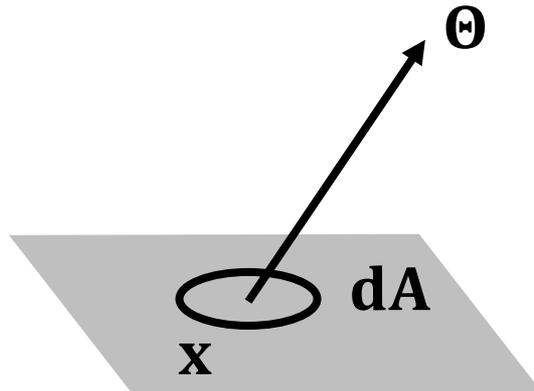
We need to find many paths…

Why Monte Carlo?

- **Radiance is hard to evaluate**

$$L_r(x \rightarrow \Theta) = \int_\Psi L(x \leftarrow \Psi) f_r(x, \Psi \rightarrow \Theta) \cos \theta_x dw_\Psi,$$

Ψ L_r

Θ

- **Sample many paths**
  - **Integrate over all incoming directions**
- **Analytical integration is difficult**
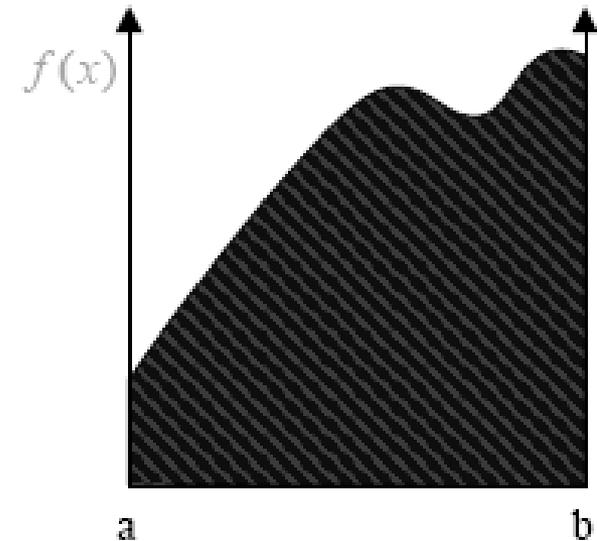 - **Need numerical techniques**

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Monte Carlo Integration

- **Numerical tool to evaluate integrals**
  - **Use sampling**
- **Stochastic errors**
- **Unbiased**
  - **On average, we get the right answer**

Numerical Integration
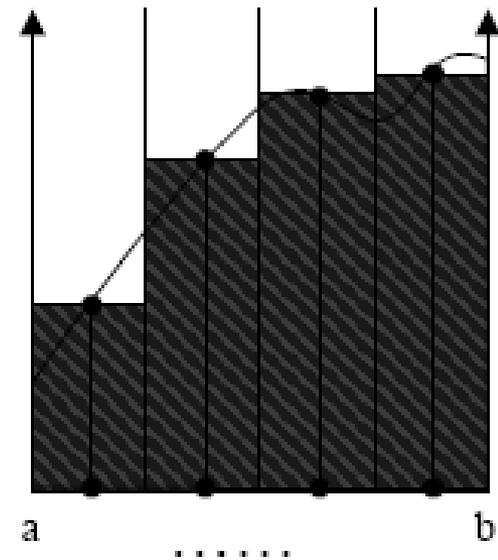
- A one-dimensional integral:

$$I = \int_a^b f(x)\,dx$$

Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x)dx$$
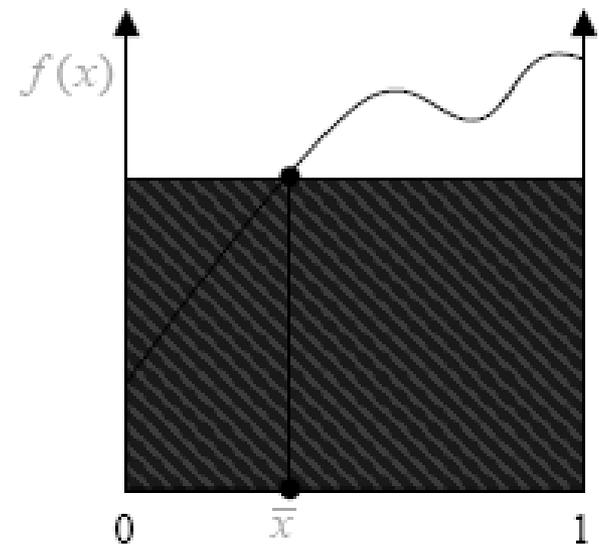
$$\approx \sum_{i=1}^{N} w_i f(x_i)$$

Monte Carlo Integration

Primary estimator:

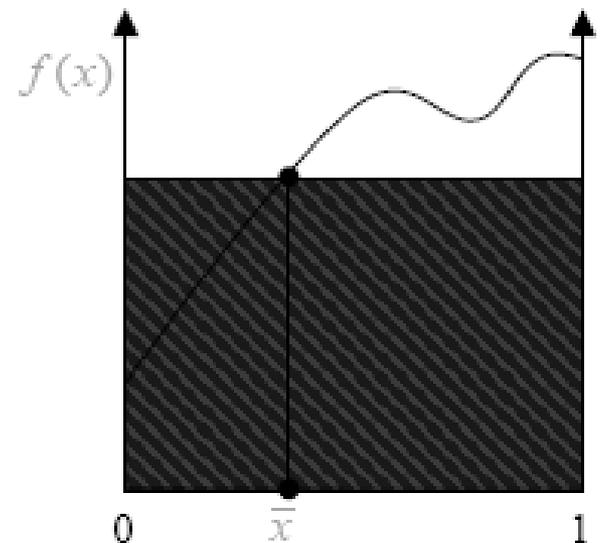$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\overline{x})$$

Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x)\,dx$$

$$I_{prim} = f(\overline{x})$$

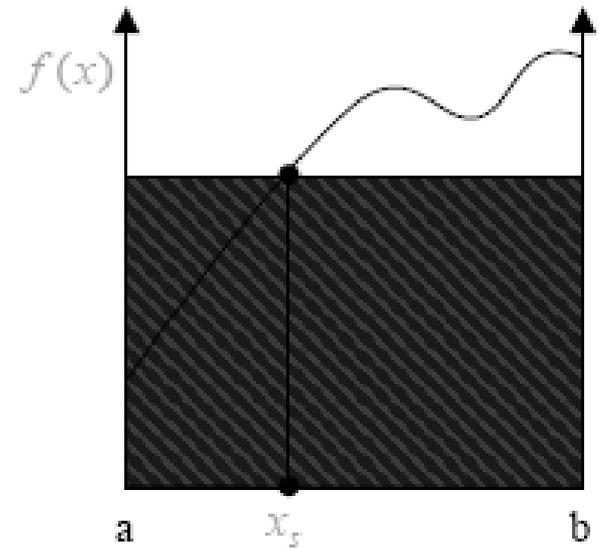$$E(I_{prim}) = \int_0^1 f(x)p(x)\,dx = \int_0^1 f(x)1\,dx = I$$

Unbiased estimator!

Monte Carlo Integration

Primary estimator:

$$I = \int\limits_a^b f(x)\,dx$$

$$I_{prim} = f(x_s)(b-a)$$
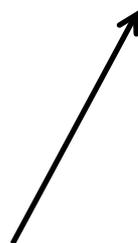

$$E(I_{prim}) = \int\limits_a^b f(x)(b-a)\,p(x)\,dx = \int\limits_a^b f(x)(b-a)\frac{1}{(b-a)}\,dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator $\rightarrow$ a measure of the stochastic error

$$\sigma^2_{prim} = \int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

- **Consider p(x) for estimate**
- **We will study it as importance sampling later**

More samples

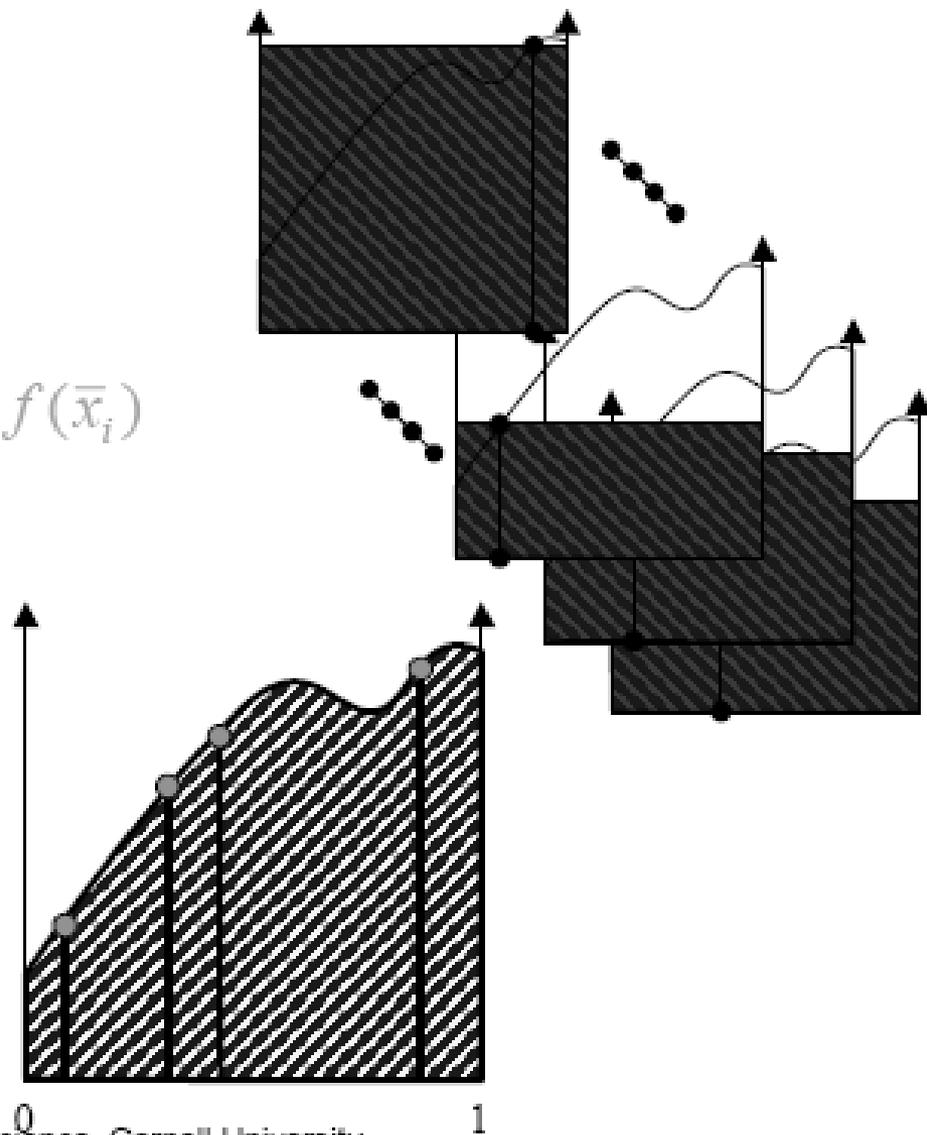## Secondary estimator

Generate N random samples x_i

Estimator: $\qquad \langle I \rangle = I_{sec} = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} f(\bar{x}_i)$

Variance $\qquad \sigma_{sec}^2 = \sigma_{prim}^2 / N$

0   1

Mean Square Error of MC Estimator

- **MSE**

$$MSE(\hat{Y}) = E[(\hat{Y} - Y)^2] = \frac{1}{N}\sum_i (\hat{Y}_i - Y_i)^2.$$

- **Decomposed into bias and variance terms**

$$MSE(\hat{Y}) = E\left[(\hat{Y} - E[\hat{Y}])^2\right] + (E(\hat{Y}) - Y)^2$$
$$= Var(\hat{Y}) + Bias(\hat{Y}, Y)^2.$$

- **Bias: how far the estimation is away from the ground truth**
- **Variance: how far the estimation is away from its average estimator**

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Bias of MC Estimator

$$E[\hat{I}] = E\left[\frac{1}{N}\sum_i \frac{f(x_i)}{p(x_i)}\right]$$

$$= \frac{1}{N}\int \sum_i \frac{f(x_i)}{p(x_i)}p(x)dx$$

$$= \frac{1}{N}\sum_i \int \frac{f(x)}{p(x)}p(x)dx, \because x_i \text{ samples have the same } p(x)$$

$$= \frac{N}{N}\int f(x)dx = I. \tag{14.6}$$

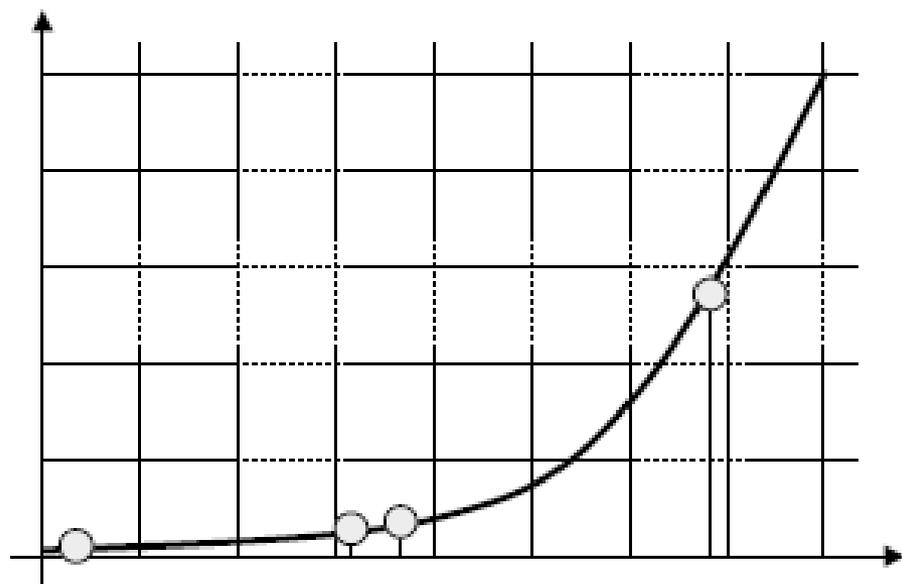- **On average, it gives the right answer: unbiased**

Variance of MC Estimator

$$Var(\hat{I}) = Var(\frac{1}{N}\sum_i \frac{f(x_i)}{p(x_i)})$$

$$= \frac{1}{N^2} Var(\sum_i \frac{f(x_i)}{p(x_i)})$$

$$= \frac{1}{N^2}\sum_i Var(\frac{f(x_i)}{p(x_i)}), \because x_i \text{ samples are independent from each other.}$$

$$= \frac{1}{N^2}NVar(\frac{f(x)}{p(x)}), \because x_i \text{ samples are from the same distribution.}$$

$$= \frac{1}{N}Var(\frac{f(x)}{p(x)}) = \frac{1}{N}\int \left(\frac{f(x)}{p(x)} - E\left[\frac{f(x)}{p(x)}\right]\right)^2 p(x)dx. \quad (14.7)$$

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MC Integration - Example

- Integral $\quad I = \int\limits_{0}^{1} 5x^4 dx = 1$

- Uniform sampling

- Samples :

$x_1 = .86 \qquad \langle I \rangle = 2.74$

$x_2 = .41 \qquad \langle I \rangle = 1.44$

$x_3 = .02 \qquad \langle I \rangle = 0.96$
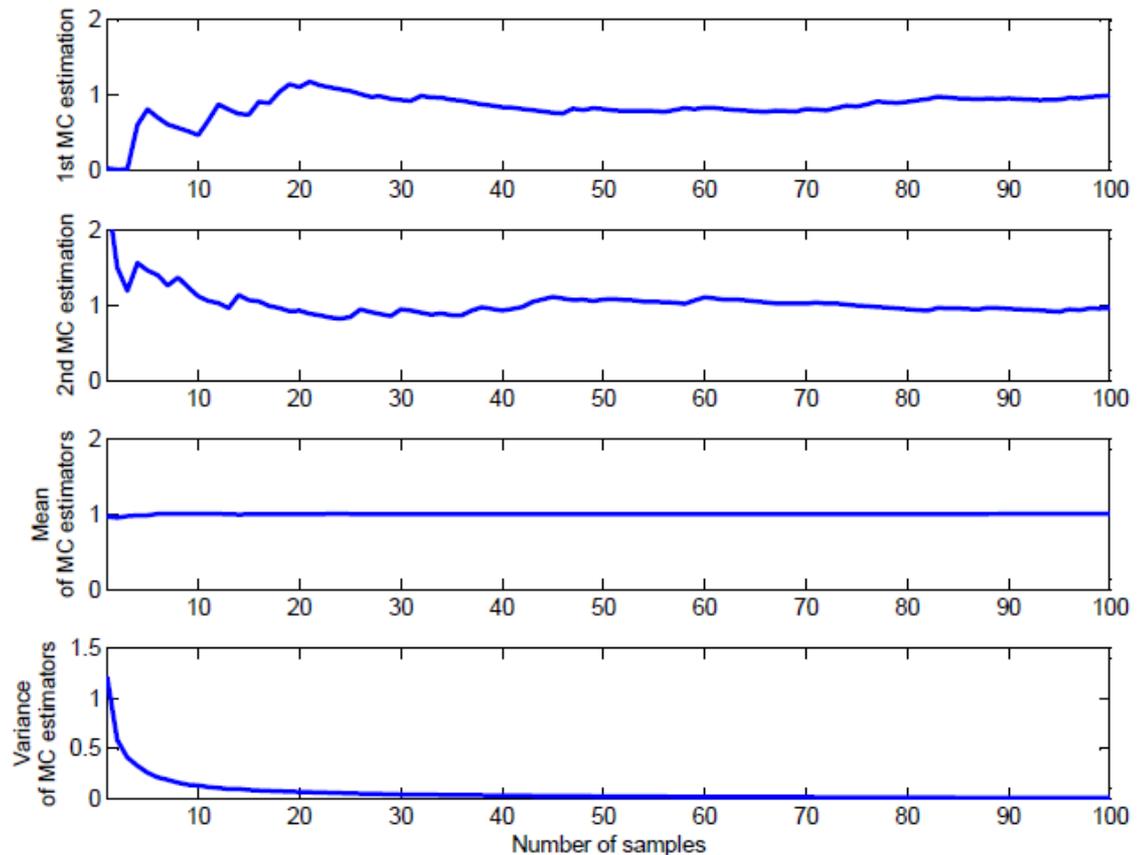
$x_4 = .38 \qquad \langle I \rangle = 0.75$

MC Integration - Example

- **Integral**

$$I = \int_0^1 4x^3 dx = 1$$
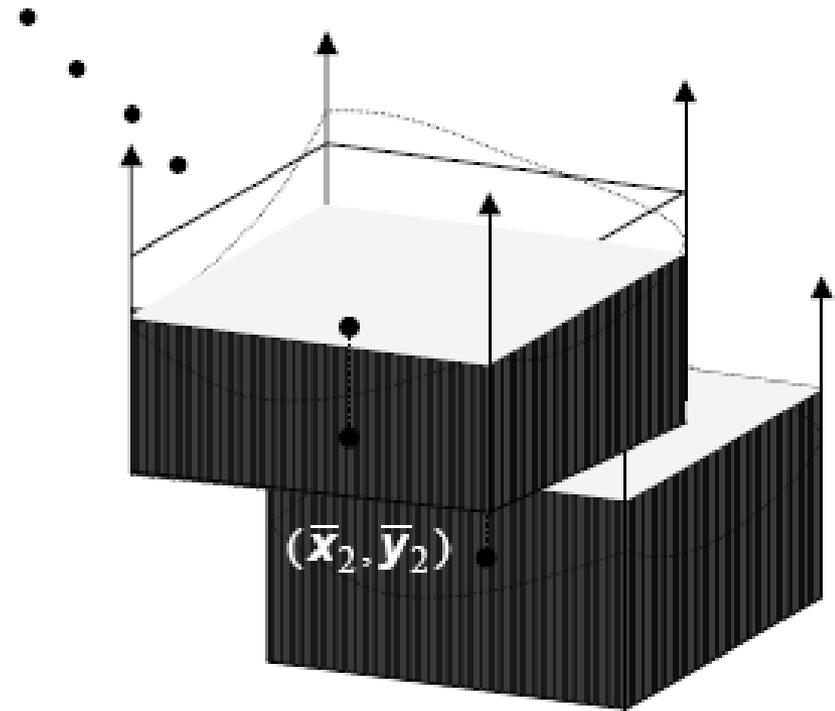
$$\hat{I} = \frac{1}{N}\sum_{i=1}^{N} 4x_i^3,$$

Code:
mc_int_ex.m

- Secondary estimator:

$$I_{sec} = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\bar{x}_i, \bar{y}_i)}{p(\bar{x}_i, \bar{y}_i)}$$
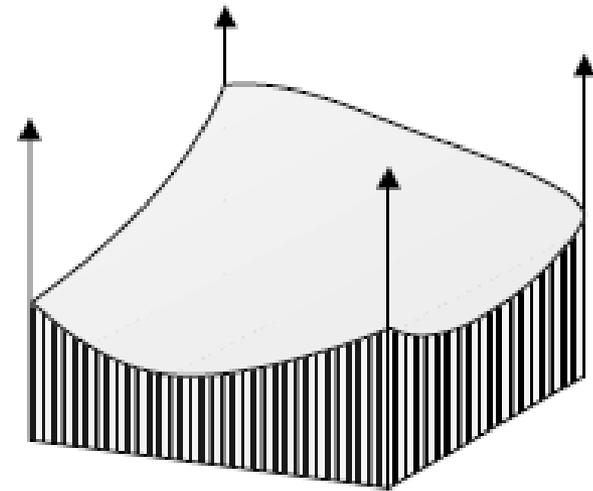

$(\bar{x}_2, \bar{y}_2)$

Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions

- Unlike quadrature

$$I = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i, y_i)}{p(x_i, y_i)}$$
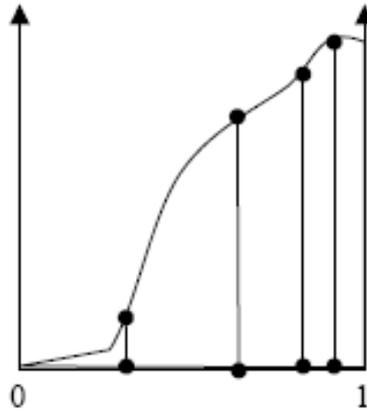
Advantages of MC

- **Convergence rate of** $O(\frac{1}{\sqrt{N}})$

- **Simple**
  - **Sampling**
  - **Point evaluation**

- **General**
  - **Works for high dimensions**
  - **Deals with discontinuities, crazy functions, etc.**

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Importance Sampling

- **Take more samples in important regions, where the function is large**

From kavita's slides

- **Sampling according to pdf (Ch. 14.4 Generating Samples)**
  - **Inverse cumulative distribution function**
  - **Rejection sampling**

Class Objectives (Ch. 14) were:

- **Sampling approach for solving the rendering equation**
  - **Monte Carlo integration**
  - **Estimator and its variance**

Next Time…

- **Monte Carlo ray tracing**

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Homework

- **Go over the next lecture slides before the class**

- **Watch two videos and submit your summaries every Mon. class**
  - **Just one paragraph for each summary**

Example:

Title: XXX XXXX XXXX

Abstract: this video is about accelerating the performance of ray tracing. To achieve its goal, they design a new technique for reordering rays, since by doing so, they can improve the ray coherence and thus improve the overall performance.

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Any Questions?

- **Submit three times before the mid-term exam**

- **Come up with one question on what we have discussed in the class and submit:**
  - **1 for already answered questions**
  - **2 for questions that have some thoughts or surprise me**

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